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Office of Naval Research
Contract N00014-75-C-0648 NR-372-012
National Science Foundation Grant - GK-31511

SYSTEM CYCLING FOR PARAMETERS ESTIMATION: OPTIMAL INPUT AND MEASUREMENT SCHEDULES DESIGNS

By
Sergio Bittanti Scida

January, 1977

Technical Report No. 666

Division of Engineering and Applied Physics
Harvard University · Cambridge, Massachusetts

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: Technical Report 666	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): System Cycling for Parameters Estimation: Optimal Input and Measurement Schedules Designs.		5. TYPE OF REPORT & PERIOD COVERED: TR-666
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Sergio Bittanti/Scida		8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0648, NSF-GK-31511
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Division of Engineering and Applied Physics, Harvard University, Cambridge, Massachusetts		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: 12 37p.		12. REPORT DATE: January 1977
		13. NUMBER OF PAGES: 37
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report): Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Parameters estimation
Optimal input design
Measurement schedules design
Periodic optimization theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper deals with the parameters estimation of a discrete-time and time-invariant system, affected by additive output noise, under periodic operation. Assuming a norm of the Fisher Information Matrix as performance index, the problems of the optimal periodic input design, of the joint optimal periodic input and measurement schedules design and of the choice of the best period for parameters estimation are formally stated. Through the introduction of the average per sample Fisher Information Matrix, they are recognized → next Page

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S/N 0102-014-6601

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as Periodic Optimal Control problems. A closed form solution is found in the particular case of linear systems with a scalar unknown parameter.

Office of Naval Research

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The research reported in this document was made possible through support extended the Division of Engineering and Applied Physics, Harvard University by the U. S. Army Research Office, the U. S. Air Force Office of Scientific Research and the U. S. Office of Naval Research under the Joint Services Electronics Program by Contracts N00014-75-C-0648 and by the National Science Foundation Grant GK 31511.

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1. INTRODUCTION

Even though experimental design has been widely studied in statistical literature, only in the last fifteen years the selection of an optimal input and measurement schedules for parameters estimation of dynamic systems came into full attention.

Since the papers by Levin [1], Levadi [2] and Nahi and Wallis [3], an increasing number of works has been devoted to the input design for identification purposes (see [4] for a survey). In [4] and [5], Mehra contributes a first systematic view of this subject by properly extending to dynamic systems the optimal experiments approach suggested by Fedorov for regression problems [6]. In most cases, the problem has been faced as an optimal control problem with a suitable norm of the Fisher Information Matrix as performance index. The motivation underlying this approach is twofold: first, the evaluation of the convariance matrix of a given estimator is often a difficult task; second, the interest in the conclusion which can be drawn through the analysis of this covariance matrix is obviously restricted to the particular kind of estimators under consideration. Therefore, it is generally assumed that an efficient estimator exists and the problem of designing an input in order to maximize a norm of the Fisher Information Matrix is directly faced.

Strangely enough, the measurement schedules design has not been paid an equally extensive attention as the input design problem. Considerable results concerning both the measurement schedules and the sensor design problems have been recently pointed out in [7] for state estimation purposes.

This paper mainly deals with the problem of jointly optimizing the input and measurement schedules for parameters estimation under system periodic operation. Actually, although the analysis of a number of periodic regimes of a system is a classical tool for the better understanding of its dynamic, only in the last few years it has been possible to provide some results concerning the input design in the frequency domain (see, for instance, [4], [8] - [10]). The present paper deals with time-invariant discrete-time dynamic systems and is organized as follows. First, a number of problems, namely, the optimal periodic input design, the joint optimal periodic input and measurement schedules design and the choice of the best period are stated in a precise formal way (Section 2). By introducing the average per sample Fisher Information Matrix, they are readily recognized (Section 3) to fall within the area of Periodic Optimization Theory. (See [11] - [13] for recent surveys on the subject and [14] for the specific tools in the discrete systems case). The particular case of linear systems affected by a scalar parameter is dealt with in Sections 4 - 6, where a closed form solution is found for most of the problems mentioned above.

2. Problem Statement

Consider the discrete-time and time-invariant dynamic system

$$x(k+1) = f(x(k),u(k),\theta) \tag{1}$$

$$y(k) = g(x(k),u(k),\theta) \tag{2}$$

with initial condition

$$x(0) = x_o(\theta), \tag{3}$$

where $u(k) \in R^m$, $x(k) \in R^n$, $y(k) \in R^p$ are the input, state and output vectors, while $\theta \in R^q$ is the unknown parameters vector. The functions $f(.,.,.)$, $g(.,.,.)$ and $x_o(\cdot)$ are assumed to be continuously differentiable with respect to their arguments and $u(\cdot)$ is regarded as a deterministic signal. Denoting by $y(k,u(\cdot), x_o(\theta),\theta)$ the solution of (1) - (3) and by $w(\cdot)$ a p-dimensional zero-mean white gaussian process with positive definite covariance matrix, viz.

$$E[w(h)w'(k)] = R\,\delta(h-k), \qquad R > 0 \ ^{(1)}$$

$$\delta(r) \triangleq \begin{cases} 0, & r \neq 0 \\ 1, & r=0, \end{cases}$$

assume that the measured output $z(\cdot)$ is given by

$$z(k) = y(k,u(\cdot),x_o(\hat{\theta}),\hat{\theta}) + w(k)$$

where $\hat{\theta}$ is the "true" value of the parameters vector.

Then, it is well known [15] that the Fisher Information Matrix can be usefully written as a function of the output sensitivity coefficients (see the pioneer work of Slepian [16] for the non-dynamic case). More precisely, denoting by $\mathscr{S}$ any discrete time set in which output observations may be performed and letting

$$\eta_i(k,u(\cdot),x_o(\theta),\theta) \triangleq \frac{\partial}{\partial\theta_i}\, y(k,u(\cdot),x_o(\theta,),\theta)$$

---

(1) Given two square matrices H and K of the same order, $H \geq K$ [$H > K$] means that $H-K$ is a semi-definite [definite] positive matrix.

$$\eta(k,u(\cdot),x_o(\theta),\theta) \triangleq | \; \eta_1(k,u(\cdot),x_o(\theta),\theta) \;\vdots\; \eta_2(k,u(\cdot),x_2(\theta),\theta) \;\vdots\; \ldots \;\vdots\; \eta_q(k,u(\cdot),x_o(\theta),\theta) \, |, \tag{4}$$

the covariance matrix C of any unbiased parameters estimator with measurement schedules $\mathscr{S}$ satisfies the inequality

$$C \geq \mathscr{M}^{-1}(\mathscr{S},\hat{\theta},u(\cdot)) \tag{5}$$

where

$$\mathscr{M}(\mathscr{S},\theta,u(\cdot)) \triangleq \sum_{k \in \mathscr{S}} \eta'(k,u(\cdot),x_o(\theta),\theta)R^{-1}\eta(k,u(.),x_o(\theta),\theta) \tag{6}$$

is the so called Fisher Information Matrix (if q=1, (5) becomes the celebrated Frechet-Cramer-Rao inequality [17]). Note that vector (4) can be regarded as the output of the sensitivity system associated to (1), (2), i.e. as the output of the $(q+1)\,n$ -th order system

$$x(k+1) = f(x(k),u(k),\theta) \tag{7}$$

$$\sigma_i(k+1) = f_x(x(k),u(k),\theta)\sigma_i(k) + f_{\theta_i}(x(k),u(k),\theta), \qquad i=1,2,\ldots,q \tag{8}$$

$$\eta_i(k) = g_x(x(k),u(k),\theta)\sigma_i(k) + g_{\theta_i}(x(k),u(k),\theta), \qquad i=1,2,\ldots,q \tag{9}$$

subject to the initial condition

$$x(0) = x_o(\theta) \tag{10}$$

$$\sigma_i(0) = \frac{\partial}{\partial\theta_i}x_o(\theta), \qquad i=1,2,\ldots,q. \tag{11}$$

Since, in this paper, only the periodic regimes of system (1), (2) are taken into consideration, condition (10) must be replaced by

$$x(0,\theta) = x(N,\theta) \tag{12}$$

where the period N is any positive integer; correspondingly, constraint (11) takes on the form

$$\sigma_i(0,\theta) = \sigma_i(N,\theta), \qquad i=1,2,\ldots,q\,. \tag{13}$$

Denoting by $\tilde{\eta}(N,\cdot,\theta)$ the periodic extension of the solution of system (7)-(9) under the periodicity constraints (12), (13) and by $\tilde{\mathcal{M}}(N,\mathcal{S},\theta,u(\cdot))$ the corresponding Fisher Information Matrix, (6) implies that

$$\tilde{\mathcal{M}}(N,\mathcal{S},\theta,u(\cdot)) = \sum_{k\in\mathcal{S}} \tilde{\eta}'(N,k,\theta)R^{-1}\tilde{\eta}(N,k,\theta) \tag{14}$$

and inequality (5) becomes

$$\tilde{C} \geq \tilde{\mathcal{M}}^{-1}(N,\mathcal{S},\hat{\theta},u(\cdot)), \tag{15}$$

where $\tilde{C}$ is the covariance matrix of any unbiased estimator based on N-periodic regimes data corresponding to the measurement schedules set $\mathcal{S}$.

In dealing with the input and measurement schedules design problem on the basis of the Fisher Information Matrix analysis, a main problem arises in connection with the necessity to evaluate this matrix at the (unknown) "true" value $\hat{\theta}$ of the parameters vector in order that (15) holds. To circumvent this difficulty, two approaches have been proposed in the literature; the first one consists in assuming that an a priori probability distribution $h(\cdot)$: $R^q \rightarrow R^1$ for the parameters vector θ is given (Bayesian approach); the second one consists in admitting that a "nominal" value $\bar{\theta}$ of θ is available (see [3], [4] and [18] for a more detailed discussion). In order to apply the latter approach to the problem considered here, the following result is needed, the proof of which is omitted as a straightforward estention to the discrete-time case of the proof given in ([19], page 253).

Proposition 1:

If

(i) system (1) admits a unique N-periodic solution $\tilde{x}(N,.,\bar{\theta})$ corresponding to the value $\theta = \bar{\theta}$ of the parameters vector

(ii) the linear N-periodic system

$$\delta x(k+1) = f_x(\tilde{x}(N,k,\bar{\theta}),u(k),\bar{\theta})\delta x(k)$$

admits no N-periodic solution other than the trivial one, i.e. the corresponding monodromy matrix has no eigenvalue equal to one,

then there exists a neighborhood Θ of θ such that, for any any $\theta \in \Theta$ system (1) admits one and only one N-periodic solution $\tilde{x}(N,.,\theta)$, given by

$$\tilde{x}(N,k,\theta) = x(N,k,\bar{\theta}) + \sum_{1}^{q}{}_{i}\ \tilde{\sigma}_i(N,k,\bar{\theta})\delta\theta_i + o(\delta\theta)$$

where $\{\tilde{\sigma}_i(N,.,\bar{\theta}),\ i=1,2,...,q,\}$ is the unique N-periodic solution of (7), (8); $\delta\theta_i \triangleq \theta_i - \bar{\theta}_i$ and

$$\lim_{||\delta\theta||\to 0} \frac{||o(\delta\theta)||}{||\delta\theta||} = 0 \cdot$$

$\square$

Roughly, this proposition says that, if $\hat{\theta} \in \Theta$, it makes sense to consider $\tilde{\mathcal{M}}(N, \mathcal{S}, \bar{\theta}, u(\cdot))$ as a fair approximation of $\tilde{\mathcal{M}}(N, \mathcal{S}, \hat{\theta}, u(.))$. In this connection, it is worthwhile to note that a similar expression of the Fisher Information Matrix has been derived in [4], with $\bar{\theta}$ playing the role of the mean of the a priori distribution of θ (Bayesian approach), while, in [18], a rather obvious iterative procedure is recommended for the approximation of $\mathcal{M}(\mathcal{S}, \hat{\theta}, u(.))$.

Denoting by R^+ the non-negative real numbers, let $\mu[\cdot]$: $R^{qxq} \to R^+$ be a suitable mapping such that $\mu[\tilde{\mathscr{M}}(N, \mathscr{S}, \bar{\theta}, u(.))]$ characterizes, in a scalar form, the "magnitude" of $\tilde{\mathscr{M}}(N, \mathscr{S}, \bar{\theta}, u(.))$ (for a discussion about some of the possible choices of $\mu[\cdot]$, see, for instance, [6]). Then, the above considerations lead to the problem of maximizing

$$L(N, \mathscr{S}, u(\cdot) \triangleq \mu[\tilde{\mathscr{M}}(N, \mathscr{S}, \bar{\theta}, u(\cdot)] \tag{16}$$

under some input constraint $u(\cdot) \in \Omega$, where Ω is a suitably specified function set, e.g.

$$\Omega \triangleq \{u(\cdot): \quad \sum_{k=o}^{N-1} v(u(k)) \leq c\} \tag{17}$$

where $c > o$ is given as well as the non-negative function $v(\cdot)$.

The analysis of the performance index (16) suggests to take into consideration the following problems as particularly significant. Let $\mathscr{Z}$ be the integers, $\mathscr{Z}^+$ the positive integers, $\mathscr{P}(\cdot)$ the power set and $\mathscr{P}_\nu(\cdot)$ the set of all the subsets of cordinality $\nu \in \mathscr{Z}^+$.

<u>Problem A</u> (Optimal Input Design)

For a given $N \in \mathscr{Z}^+$ and $\mathscr{S} \in \mathscr{P}(\mathscr{Z})$, find $u_o(\cdot) \in \Omega$ such that

$$L(N, \mathscr{S}, u_o(\cdot)) \geq L(N, \mathscr{S}, u(\cdot)) \quad \forall u(\cdot) \in \Omega \quad .$$

<u>Problem B</u> (Optimal Input and Period Design)

For a given $\mathscr{S} \in \mathscr{P}(\mathscr{Z})$, find $\underline{N_\theta} \in \mathscr{Z}^+$ and $\underline{u_\theta}(\cdot) \in \Omega$ such that

$$L(\underline{N_\theta}, \mathscr{S}, \underline{u_\theta}(\cdot)) \geq L(N, \mathscr{S}, u(\cdot)), \quad \forall (N, u(\cdot)) \in \mathscr{Z}^+ x\Omega \quad .$$

<u>Problem C</u> (Optimal Input and Measurement Allocation Design)

For a given pair $(N, \nu) \in \mathscr{Z}^+ \times \mathscr{Z}^+$, find $\mathscr{S}_\phi \in \mathscr{P}_\nu(\mathscr{Z})$ and $u_\phi(\cdot) \in \Omega$ such that

$$L(N, \mathscr{S}_\phi, u_\phi(.)) \geq L(N, \mathscr{S}, u(.)), \qquad \forall(\mathscr{S}, u(.)) \in \mathscr{P}_\nu(\mathcal{Z}) x \Omega.$$

□

Since the computational complexity of the parameter estimator strongly depends on the cardinality of $\mathscr{S}$, it is obvious that a first meaningful choice of a good menasurement schedule set is among "computationally equivalent sets", namely sets of given cardinality (Problem C). In order to state in a consistent way a more general optimal measurement schedule design problem it is necessary to introduce a suitable measure of the obtainable information per unit of computational effort.

Problem D (Optimal Input and Measurement Schedule Design)

For a given $N \in \mathcal{Z}^+$, find $\mathscr{S}_\phi \in \mathscr{P}(\mathcal{Z})$ and $u_\phi(.) \in \Omega$ such that

$$\frac{1}{|\mathscr{S}_\phi|} L(N, \mathscr{S}_\phi, u_\phi(.)) \geq \frac{1}{|\mathscr{S}|} L(N, \mathscr{S}, u(.)), \qquad \forall(\mathscr{S}, u(.)) \in \mathscr{P}(\mathcal{Z}) x \Omega,$$

where $|.|$ denotes the cardinality.

□

The analysis of Problem D directly leads to the introduction of an average per sample Fisher Information Matrix and constitutes the subject of the next Section 3.

Finally, the formal statement of an optimal input, measurement schedule and period design problem is omitted as trivial.

3. AVERAGE PER SAMPLE FISHER INFORMATION MATRIX UNDER PERIODIC OPERATION

Considering any finite measurement schedule $\mathscr{S}$, the average per sample Fisher Information Matrix under periodic operation is defined as follows:

$$M(N, \mathscr{S}, u(\cdot)) \triangleq \frac{1}{|\mathscr{S}|} \mathscr{M}(N, \mathscr{S}, \bar{\theta}, u(.)) \tag{18}$$

(in [4], under stability assumptions - here not necessary - the terminology "asymptotic per sample Fisher Information Matrix" is adopted). In analogy with (16), the scalar

$$J(N, \mathscr{S}, u(\cdot)) \triangleq \mu[M(N, \mathscr{S}, u(\cdot))] \tag{19}$$

is associated to matrix (19).

Letting

$$S_N = [0, N-1]$$

$$\Omega_j(P) \triangleq \{k : k-jN \in P \mid P \in \mathscr{P}(S_N),\ j \in \mathscr{Z}\}$$

$$\tilde{\mathscr{P}}(\mathscr{Z}) \triangleq \{\{\bigcup_{j \in J} \Omega_j(P)\} \mid P \in \mathscr{P}(S_N),\ J \in \mathscr{P}(\mathscr{Z})\} \tag{20}$$

in this section, the families of matrices

$$\mathscr{T}_{S_N} \triangleq \{M(N, \mathscr{S}, u(\cdot)) \mid \mathscr{S} \in \mathscr{P}(S_N)\}$$

$$\tilde{\mathscr{T}} \triangleq \{M(N, \mathscr{S}, u(\cdot)) \mid \mathscr{S} \in \tilde{\mathscr{P}}(\mathscr{Z})\}$$

$$\mathscr{T} \triangleq \{M(N, \mathscr{S}, u(\cdot)) \mid \mathscr{S} \in \mathscr{P}(\mathscr{Z})\}$$

and the corresponding sets of scalars given by (19) are usefully compared.

Given $h \in \mathscr{Z}$ and a measurement schedule $\mathscr{S} \in \mathscr{P}(\mathscr{Z})$, denote by

$$\mathscr{S}_h \triangleq \mathscr{S} \cap [hN, (h+1)N-1]$$

$$H \triangleq \{h \mid \mathscr{S}_h \neq \phi\}.$$

Then, the following preliminary result can be stated.

<u>Proposition 2</u>

The average per sample Fisher Information Matrix corresponding to any measurement schedule $\mathscr{S} \in \mathscr{P}(\mathscr{Z})$ can be given the form

$$M(N,\mathscr{S},u(\cdot)) = \sum_{h\,\in\,H} \frac{|\mathscr{S}_h|}{|\mathscr{S}|}\; M(N,\mathscr{S}_h,u(\cdot)). \tag{21}$$

Proof:

Equations (14) and (18) immediately imply (21). $\square$

Noticing that $\mathscr{T}_{S_N}$ is a finite set, denote by χ its cardinality and by

$$CH[\mathscr{T}_{S_N}] \triangleq \{ \sum_{1}^{\chi} \alpha_i K_i \,|\, K_i \in \mathscr{T}_{S_N},\quad \forall i=1,2,\dots,\chi$$

$$\alpha_i \in R^+,\ \forall i=1,2,\dots,\chi;\ \sum_{1}^{\chi} \alpha_i = 1\}$$

its convex hull; furthermore, let

$$W \triangleq \{ \frac{a}{b} \mid 0 \le a \le N;\ b \ge N;\ (a,b) \in \mathscr{Z}\times\mathscr{Z} \}$$

and define

$$CW[\mathscr{T}_{S_N}] \triangleq \{ \sum_{1}^{\chi} w_i K_i \mid K_i \in \mathscr{T}_{S_N},\ \forall i=1,2,\dots,\chi;$$

$$w_i \in W,\ \forall i=1,2,\dots,\chi;\ \sum_{1}^{\chi} w_i = 1\}.$$

Corollary 1

For a given pair $(N,u(\cdot)) \in \mathscr{Z}^+ \times \Omega$, it is

$$\mathscr{T} = CW[\mathscr{T}_{S_N}], \tag{22}$$

and

$$\tilde{\mathscr{T}} = \mathscr{T}_{S_N}. \tag{23}$$

Proof

Let

$$\hat{\mathscr{S}}_h \triangleq \{k \in S_N \mid k \ (\mathrm{mod}\ N) \in \mathscr{S}_h\}.$$

Then, in view of the periodicity of $\tilde{\eta}(N,.,\bar{\theta})$, Eq. (21) can be written as follows

$$M(N,\hat{\mathscr{P}},u(\cdot)) = \sum_{h\in H} \frac{|\hat{\mathscr{P}}_h|}{|\mathscr{P}|} M(N,\hat{\mathscr{P}}_h,u(\cdot)), \tag{24}$$

so that the first part of the Corollary easily follows.

As for the second part, noticing that, if $\mathscr{P}\in\boldsymbol{\mathscr{P}}(\mathscr{Z})$, then $\hat{\mathscr{P}}_{h_1} = \hat{\mathscr{P}}_{h_2}$, $\forall(h_1,h_2)\in H\times H$, Eq. (24) implies that

$$\tilde{\mathscr{T}} \subseteq \mathscr{T}_{S_N}.$$

Since the converse of this inclusion is trivial, statement (23) is proved.

<u>Theorem 1</u>

For a given pair $(N,u(\cdot))\in\mathscr{Z}^{+}\times\Omega$, the maximum of $J(N,\mathscr{P},u(\cdot))$ with respect to $\mathscr{P}\in\tilde{\boldsymbol{\mathscr{P}}}(\mathscr{Z})$ exists and is given by

$$\max_{\mathscr{P}\in\tilde{\boldsymbol{\mathscr{P}}}(\mathscr{Z})} J(N,\mathscr{P},u(\cdot)) = \max_{\mathscr{P}\in\boldsymbol{\mathscr{P}}(S_N)} J(N,\mathscr{P},u(\cdot)). \tag{25}$$

Furthermore, if $\mu[\cdot]$ is a convex mapping (in particular a matrix norm), the maximum of $J(N,\mathscr{P},u(\cdot))$ with respect to $\mathscr{P}\in\boldsymbol{\mathscr{P}}(\mathscr{Z})$ exists and is given by

$$\max_{\mathscr{P}\in\boldsymbol{\mathscr{P}}(\mathscr{Z})} J(N,\mathscr{P},u(\cdot)) = \max_{\mathscr{P}\in\boldsymbol{\mathscr{P}}(S_N)} J(N,\mathscr{P},u(\cdot)).$$

<u>Proof</u>

Since $\boldsymbol{\mathscr{P}}(S_N)$ is a finite set, the maximum of $J(N,\mathscr{P},u(\cdot))$ with respect to $\mathscr{P}\in\boldsymbol{\mathscr{P}}(S_N)$ exists. From the obvious equalities

$$\max_{\mathscr{S} \in \mathscr{P}(S_N)} J(N, \mathscr{S}, u(\cdot)) = \max_{K \in \mathscr{T}_{S_N}} \mu[K] \tag{26}$$

$$\max_{\mathscr{S} \in \tilde{\mathscr{P}}(\mathscr{Z})} J(N, \mathscr{S}, u(\cdot)) = \max_{K \in \tilde{\mathscr{T}}} \mu[K],$$

statement (23) immediately imply equality (25). If $\mu[\cdot]$ is a convex mapping, the maximum of $\mu[\cdot]$ over the polyhedral convex set $CH[\mathscr{T}_{S_N}]$ exists and is given by

$$\max_{K \in CH[\mathscr{T}_{S_N}]} \mu[K] = \max_{K \in \mathscr{T}_{S_N}} \mu[K]. \tag{27}$$

Then, the trivial inclusions

$$\mathscr{T}_{S_N} \subseteq CW[\mathscr{T}_{S_N}] \subseteq CH[\mathscr{T}_{S_N}],$$

together with equality (27), implies that the maximum of $\mu[.]$ over $CW[\mathscr{T}_{S_N}]$ exists too and is given by

$$\max_{K \in CW[\mathscr{T}_{S_N}]} \mu[K] = \max_{K \in \mathscr{T}_{S_N}} \mu[K]. \tag{28}$$

From (22) and (26) - (28), the second part of the theorem follows. $\square$

The above results suggest to replace Problems A, B, D with the following problems, which are stated in much simpler terms by making reference to the time interval [0, N-1] only.

<u>Problem $\tilde{A}$</u>

For a given $N \in \mathscr{Z}^+$ and $\mathscr{S} \in \mathscr{P}(S_N)$, find $u^o(\cdot) \in \Omega$ such that

$$J(N, \mathscr{S}, u^o(\cdot)) \geq J(N, \mathscr{S}, u(\cdot)), \quad \forall u(\cdot) \in \Omega.$$

Problem $\tilde{B}$

For a given $\mathscr{S} \in \mathscr{P}(S_N)$, find $N^{\circ} \in \mathcal{Z}^+$ and $u^{\circ}(\cdot) \in \Omega$ such that

$$J(N^{\circ},\mathscr{S},u^{\circ}(\cdot)) \geq J(N,\mathscr{S},u(\cdot)), \quad \forall (N,u(\cdot)) \in \mathcal{Z}^+ x\Omega.$$

Problem $\tilde{D}$

For a given $N \in \mathcal{Z}^+$, find $\mathscr{S}^{\phi} \in \mathscr{P}(S_N)$ and $u^{\phi}(\cdot) \in \Omega$ such that

$$J(N,\mathscr{S}^{\phi},u^{\phi}(\cdot)) \geq J(N,\mathscr{S},u(\cdot)), \quad \forall (\mathscr{S},u(\cdot)) \in \mathscr{P}(S_N) x\Omega \quad . \quad \square$$

From a conceptual point of view, it is worthwhile to define also the following problem.

Problem $\tilde{C}$

For a given $N \in \mathcal{Z}^+$ and $\nu \in [1,N]$, find $\mathscr{S}^{\phi} \in \mathscr{P}_{\nu}(S_N)$ and $u^{\phi}(\cdot) \in \Omega$ such that

$$J(N,\mathscr{S}^{\phi},u^{\phi}(\cdot)) \geq J(N,\mathscr{S},u(\cdot)), \quad \forall (\mathscr{S},u(\cdot)) \in \mathscr{P}_{\nu}(S_N) x\Omega \quad . \quad \square$$

However, note that it is mainly the knowledge of the solution of Problem $\tilde{D}$ which does provide useful indications for the search of the solution of Problem C.

Problems $\tilde{A}$ - $\tilde{D}$ are suited to be faced by general Periodic Optimization techniques (see, for instance, [14]); however, it is only in the particularly important case of linear systems that these problems can actually be worked our further up to obtain results which can eventually be exploited in a simple and more direct way. In order to illustrate this point, the remainder of the paper is entirely devoted to the analysis of the special case of linear systems affected by a scalar parameter; as a matter of fact, most of the problems can in this case be given a closed form solution.

4. OPTIMAL INPUT DESIGN

Consider system (1), (2) with

$$f(x(k),u(k),\theta) = A(\theta)\, x(k) + B(\theta)u(k) \tag{29}$$

$$g(x(k),u(k),\theta) = C(\theta)x(k), \tag{30}$$

where $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are continuously differentiable matrices of suitable order and θ is a scalar parameter. In order that, for each N, the corresponding $2n$ -th order sensitivity system admits a unique N-periodic solution in a neighborhood of the parameter nominal value $\bar{\theta}$, it is assumed that no eigenvalue of $A(\bar{\theta})$ has unitary absolute value.

Since, in the here considered case, the Fisher Information Matrix is obviously a (non negative) scalar, in view of (14), (18), and (19), the performance index can be given the form

$$J(N,\mathscr{S},u(\cdot)) = \frac{1}{|\mathscr{S}|} \sum_{k \in \mathscr{S}} \tilde{\eta}'(N,k,\bar{\theta})\; R^{-1}\tilde{\eta}(N,k,\bar{\theta}), \tag{31}$$

where $\mathscr{S} \in \mathscr{P}(S_N)$.

As far as constraint (17) is concerned, a power-constraint of the form

$$\frac{1}{N} \sum_{k \in S_N} u'(k)Qu(k) \leq c \tag{32}$$

is now considerd, where Q is any given positive definite matrix. Without any loss of generality, constraint (32) will be given the simpler form

$$\frac{1}{N} \sum_{k \in S_N} u'(k)u(k) \leq 1 \tag{33}$$

In order to synthetically state the following results, some preliminary notations and definitions are now introduced.

Let

$$G(z) \triangleq C_\theta(\bar{\theta})(zI-A(\bar{\theta}))^{-1}B(\bar{\theta})$$

$$+ C(\bar{\theta})(zI-A(\bar{\theta}))^{-1}A_\theta(\bar{\theta})(zI-A(\bar{\theta}))^{-1}B(\bar{\theta})$$

$$+ C(\bar{\theta})(zI-A(\bar{\theta}))^{-1}B_\theta(\bar{\theta})$$

be the z-transfer matrix of system (7) - (9), (29), (30) corresponding to the parameter nominal value and $\Pi_{r,s}(N,\mathscr{S})$ be the following mxm complex valued matrix

$$\Pi_{r,s}(N,\mathscr{S}) = b_{r,s}(N,\mathscr{S})G^*(\exp(jr\,\frac{2\pi}{r}))\;R^{-1}G(\exp(js\,\frac{2\pi}{N})),$$

$$(r,s) \in S_N x S_N, \tag{34}$$

where the scalar $b_{r,s}(N,\mathscr{S})$ is given by

$$b_{r,s}(N,\mathscr{S}) \triangleq \frac{1}{|\mathscr{S}|}\sum_{k\in\mathscr{S}} \exp(-jrk\,\frac{2\pi}{N})\;\exp(jsk\,\frac{2\pi}{N}) \tag{35}$$

and * denotes conjugate transpose (in the following, the trivial case when $G(z) = 0$, $\forall z$, is not considered). Since

$$\Pi^*_{r,s}(N,\mathscr{S}) = \Pi_{s,r}(N,\mathscr{S}), \quad \forall(r,s) \in S_N x S_N$$

the mNxmN matrix

$$\Gamma(N,\mathscr{S}) \triangleq [\Pi_{r,s}(N,\mathscr{S})]_{\substack{r=0,1,\dots,N-1\\ s=0,1,\dots,N-1}} \tag{36}$$

turns out to be Hermitian.

Finally, consider the Finite Fourier Transform expansion [20] of $u(\cdot)$

$$u(k) = \sum_{h\in S_N} U_h \exp(jkh\,\frac{2\pi}{N}) \tag{37}$$

and let

$$U^* = [U_o^* \; U_1^* \; \dots \; U_{N-1}^*]. \tag{38}$$

Theorem 2

The performance index (31), under (7) - (9), (12), (13), (29), (30) can be given the form

$$J(N, \mathscr{S}, u(\cdot)) = U^* \Gamma(N, \mathscr{S}) U \tag{39}$$

while constraint (33) can be written as

$$U^* U \leq 1. \tag{40}$$

Proof

When $\theta = \bar{\theta}$, the output of the sensitivity system (7) - (9), (29), (30) under (12), (13), corresponding to the input (37) is given by

$$\tilde{\eta}(N,k,\bar{\theta}) = \sum_{h \in S_N} G(\exp(jh \tfrac{2\pi}{N}))\; U_h \; \exp(jkh \tfrac{2\pi}{N}) \tag{41}$$

From (31) and (41), it follows that

$$J(N, \mathscr{S}, u(\cdot)) = \sum_{r \in S_N} \sum_{s \in S_N} U_r^* \Pi_{r,s}(N, \mathscr{S}) U_s,$$

which is equivalent to (39); (40) is direct consequence of (33) and (37).

Remark 1

The matrix $\Gamma(N, \mathscr{S})$ is obviously positive semi-definite. Since it is also Hermitian, its eigenvalues are all non-negative real numbers. Furthermore, it can easily be seen, from (34) and (35), that

$$\Pi_{N-r,\, N-s}(N, \mathscr{S}) = \overline{\Pi_{r,s}(N, \mathscr{S})}, \quad \forall (r,s) \in S_N x S_N, \quad \forall \mathscr{S} \in \mathscr{P}(S_N)$$

($\overline{}$ denotes conjugate). Therefore, to each eigenvalue λ of $\Gamma(N, \mathscr{S})$ it can be associated a (non zero) eigenvector U^λ with the property

$$U^\lambda_{N-h} = \bar{U}^\lambda_h, \quad \forall h \in S_N. \tag{42}$$

□

The solution of Problem $\tilde{A}$ can now be obtained. Let

$$F(N) \triangleq [f_{r,s}]_{\substack{r=0,1,\dots,N-1 \\ s=0,1,\dots,N-1}}$$

$$f_{r,s} \triangleq \exp(jrs \frac{2\pi}{N})$$

and denote by

$$\mathscr{F}(N) \triangleq F(N) x I_m$$

the Kronecker product (see [21], page 227) of F(N) and the m-th order identity matrix I_m. Furthermore, define the mN vector $\mathscr{U}$ as

$$\mathscr{U}' \triangleq [u'(0)\ u'(1)\ \dots\ u'(N-1)] \tag{43}$$

and denote by $\mathscr{U}^o(N,\mathscr{S})$ vector (43) corresponding to the *solution* $u^o(N,\mathscr{S},\cdot)$ of Problem $\tilde{A}$, where, for the sake of notational clarification, the dependence on N and $\mathscr{S}$ is explicity pointed out.

Theorem 3

Let $U^o(N,\mathscr{S})$ be a unitary norm eigenvector of $\Gamma(N,\mathscr{S})$ corresponding to its maximum eigenvalue $\lambda_{max}[\Gamma(N,\mathscr{S})]$ and satisfying condition (42). Then,

$$\mathscr{U}^o(N,\mathscr{S}) = \mathscr{F}(N)\ U^o(N,\mathscr{S}) \tag{44}$$

and

$$J(N, \mathscr{S}, u^o(N,\mathscr{S},\cdot)) = \lambda_{max}[\Gamma(N,\mathscr{S})].$$

Proof

Since $\Gamma(N,\mathscr{S})$ is a positive semi-definite Hermitian matrix, it is well known that the maximum of (39) under constraint (40) is reached when U is a unitary norm eigenvector of $\Gamma(N,\mathscr{S})$ associated to its maximum eigenvalue. On the other hand, in order that the inverse transform $\mathscr{U}$ of U (given by $\mathscr{U} = \mathscr{F}(N)U$) be a real vector, it is necessary and sufficient that condition (42) is

satisfied (see Remark 1). $\square$

As far as the particularly significant cases $\mathscr{S} = S_N$ and $\mathscr{S} = S_1 \triangleq \{0\}$ are concerned, the following results are easily derived from Theorem 3.

Theorem 4 ($\mathscr{S} = S_N$)

Let

$$\Pi(z) \triangleq G^*(z)R^{-1}G(z) \tag{45}$$

and denote by V_h a unitary norm eigenvector of $\Pi(\exp(jh\,\frac{2\pi}{N}))$ corresponding to its maximum eigenvalue λ_h. Then, if $h^o \in S_N$ is defined as follows

$$\lambda_{h^o} \geq \lambda_h, \qquad \forall h \in S_N,$$

the optimal input is given by

$$u^o(N, S_N, k) = \begin{cases} \sqrt{2}\,\mathrm{Re}\{V_{h^o} \exp(jkh^o\,\frac{2\pi}{N})\}, & h^o \neq 0 \quad (46.a) \\ \\ V_0 \qquad\qquad , & h^o = 0 \quad (46.b) \end{cases}$$

(Re $\{.\}$ denotes the real part).

Proof

If $\mathscr{S} = S_N$, (34) and (35) imply that

$$\Pi_{r,s}(N, S_N) = \begin{cases} \Pi(\exp(jr\,\frac{2\pi}{N})) \quad , & s=r \\ \\ 0 \quad , & s\neq r \end{cases} .$$

Consequently, $\Gamma(N,S_N)$ turns out to be a block diagonal matrix (see (36)), so that

$$\lambda_{max}[\Gamma(N,S_N)] = \lambda_{h^o}$$

and

$$U^o(N,S_N)' = \begin{cases} \frac{1}{\sqrt{2}}[\Psi'\Psi' \ldots \Psi' V'_{h^o}\Psi' \ldots \Psi'\bar{V}'_{h^o}\ \Psi' \ldots \Psi'],\ h^o \neq 0 & (47.a) \\ [V'_o\ \Psi'\ \Psi' \ldots \Psi'] \qquad ,\ h^o = 0, & (47.b) \end{cases}$$

where Ψ is the m-dimensional zero vector while V_{h^o} and $\bar{V}_{h^o}$ enter (47.a) in the h^o-th and $(N-h^o)$-th positions respectively (note that λ_{h^o} is an eigenvalue of $\Pi_{N-h^o,N-h^o}(N,S_N)$ and $\bar{V}_{h^o}$ the associated unitary norm eigenvector). From (44) and (47), the thesis follows.

Theorem 5 ($\mathscr{S} = S_1$)

Let

$$\Phi(N) \triangleq \left[G(\exp(j0\ \tfrac{2\pi}{N}))\ \middle|\ G(\exp(j1\ \tfrac{2\pi}{N}))\ \middle|\ \ldots\ \middle|\ G(\exp(j(N-1)\ \tfrac{2\pi}{N})) \right] \qquad (48)$$

$$Z \triangleq \sqrt{R^{-1}} \qquad (49)$$

$$H(N) \triangleq Z\Phi(N) \qquad (50)$$

and denote by $h_i(N)$ the i-th row of $H(N)$. Then,

$$\Gamma(N,S_1) = \textstyle\sum_{i \in [1,p]} h_i^*(N)h_i(N). \qquad (51)$$

Proof

From (35), it follows that

$$b_{r,s}(N,S_1) = 1, \qquad \forall (r,s) \in S_N x S_N.$$

Therefore, (39) becomes

$$\Gamma(N,S_1) = H^*(N)H(N) = \textstyle\sum_{i \in [1,p]} h_i^*(N)h_i(N).$$

□

As for the general case $\mathscr{S} \in \mathscr{P}(S_N)$, the following general properties of $\Gamma(N,\mathscr{S})$ can be usefully pointed out.

Theorem 6

Let

$$\rho \triangleq \textstyle\sum_{h \in S_N} \text{rank}\ [G(\exp(jh\ \tfrac{2\pi}{N}))];$$

then

$$\text{rank}\ [\Gamma(N,\mathscr{S})] = \min\{\rho, p|\mathscr{S}|\}, \qquad \mathscr{S} \in \mathscr{P}(S_N).$$

<u>Proof</u>

Letting

$$\mathscr{Z}(N) \triangleq \text{diag}[Z,Z,\ldots,Z], \qquad \mathscr{Z}(N) \in R^{pNxpN}$$

$$\mathscr{G}(N) \triangleq \text{diag}[G(\exp(jo\ \tfrac{2\pi}{N})),\ G(\exp(j1\ \tfrac{2\pi}{N}))\ ,\ldots,\ G(\exp(j(N-1)\ \tfrac{2\pi}{N})] \tag{52}$$

$$B(N,\mathscr{S}) \triangleq [b_{r,s}]_{\substack{r=0,1,\ldots,N-1\\ s=0,1,\ldots,N-1}}$$

$$\mathscr{B}(N,\mathscr{S}) \triangleq B(N,\mathscr{S})\ X\ I_p,$$

a simple but cumbersome computation shows that

$$\Gamma(N,\mathscr{S}) = \mathscr{G}^*(N)\,\mathscr{Z}^*(N)\,\mathscr{B}(N,\mathscr{S})\,\mathscr{Z}(N)\,\mathscr{G}(N),$$

so that

$$\text{rank}\ [\Gamma(N,\mathscr{S})] \leq \min\{\text{rank}[\mathscr{G}(N)],\ \text{rank}[\mathscr{B}(N,\mathscr{S})]\}.$$

Since

$$\text{rank}\ [\mathscr{B}(N,\mathscr{S})] = p\ \text{rank}[B(N,\mathscr{S})]$$

(see [21], page 227) and

$$B(N,\mathscr{S}) = \frac{1}{|\mathscr{S}|} \sum_{k\in\mathscr{S}} B^*(k)B(k),$$

where

$$B(k) \triangleq [\exp(jk0\ \tfrac{2\pi}{N}) \quad \exp(jk1\ \tfrac{2\pi}{N}) \quad \ldots \quad \exp(jk(N-1)\ \tfrac{2\pi}{N})]\ ,$$

then, if $\mathscr{S} \in \mathscr{P}(S_n)$,

$$\text{rank}\ [\mathscr{B}(N,\mathscr{S})] = p|\mathscr{S}|$$

and the theorem follows.

Corollary 2

If $N \geq np|\mathscr{S}|$, then

$$\text{rank}\,[\Gamma(N,\mathscr{S})] \leq p|\mathscr{S}|, \qquad \mathscr{S} \in \mathscr{P}(S_N).$$

Proof

Since, by assumption, G(z) is not identically equal to zero, there is at least one of its elements which can be zero at most in n-1 points of the complex plane. Therefore,

$$\sum_{h \in [0,n-1]} \text{rank}[G(\exp(jh \tfrac{2\pi}{N}))] \geq 1, \quad \forall N \geq n.$$

Proposition 3

The trace of $\Gamma(N,\mathscr{S})$ is the same for all $\mathscr{S} \in \mathscr{P}(S_N)$.

Proof

The proof is trivially obtained by noticing that, for each $r \in S_N$,

$$\Pi_{r,r}(N,\mathscr{S}) = \Pi(\exp(jr \tfrac{2\pi}{N})), \quad \forall \mathscr{S} \in \mathscr{P}(S_n).$$

5. JOINT OPTIMAL INPUT AND MEASUREMENT SCHEDULE DESIGN

The key result for the solution of Problem $\tilde{D}$ in the simplest case of linear system affected by a scalar parameter is supplied by the following theorem.

Theorem 7

For each $\mathscr{S} \in \mathscr{P}(S_N)$, the maximum eigenvalue of $\Gamma(N,\mathscr{S})$ satisfies the following inequalities

$$\frac{1}{|\mathscr{S}|} \max_{i=1,\dots,p} ||h_i(N)||^2 \leq \lambda_{max}[\Gamma(N,\mathscr{S})] \leq \lambda_{max}[\sum_{i \in [1,p]} h_i^*(N) h_i(N)], \tag{53}$$

where $h_i(N)$ is defined as in Theorem 5.

Proof

Letting

$$T(N,k) \stackrel{\Delta}{=} \mathrm{diag}[\exp(jk0\,\tfrac{2\pi}{N}),\ \exp(jk1\,\tfrac{2\pi}{N}),\ \ldots,\ \exp(jk(N-1)\,\tfrac{2\pi}{N})]$$

$$\mathscr{T}(N,k) \stackrel{\Delta}{=} T(N,k) \times I_m$$

$$v(N,k,i) \stackrel{\Delta}{=} h_i(N)\,\mathscr{T}(N,k),$$

$\Gamma(N,\mathscr{S})$ can be given the following form

$$\Gamma(N,\mathscr{S}) = \frac{1}{|\mathscr{S}|} \sum_{k \in \mathscr{S}} \sum_{i \in [1,p]} v^*(N,k,i)v(N,k,i),$$

which, in turn, implies that

$$\lambda_{\max}[\Gamma(N,\mathscr{S})] = \frac{1}{|\mathscr{S}|} \max_{||\xi||=1} \left\{ \sum_{k \in \mathscr{S}} \xi^* \mathscr{T}^*(N,k) \left[\sum_{i \in [1,p]} h_i^*(N)h_i(N)\right] \mathscr{T}(N,k)\xi \right\}. \tag{54}$$

For each $k \in S_N$, $\mathscr{T}(N,k)$ is an orthogonal matrix. Hence, if $||\xi|| = 1$,

$$\xi^* \mathscr{T}(N,k) \left[\sum_{i \in [1,p]} h_i^*(N)h_i(N)\right] \mathscr{T}(N,k)\xi \leq \lambda_{\max}\left[\sum_{i \in [1,p]} h_i^*(N)h_i(N)\right], \tag{55}$$

for all $k \in S_N$.

From (54) and (55), the right hand inequality in statement (53) follows, while the left hand side is an easy consequence of (54).

Remark 2

When $\mathscr{S} \in \tilde{\mathscr{P}}(\mathscr{Z})$, Theorem 7 yields interesting bounds for $\tilde{\mathscr{M}}^{-1}(N,\mathscr{S},\bar{\theta},u_o(\cdot))$, namely the Frechet-Cramer-Rao approximate covariance lower bound. More

precisely, assuming, without any loss of generality, that $\mathscr{S}_1 \triangleq \mathscr{S} \cap S_N \neq \phi$, (14), (31) and (53) imply that

$$\frac{1}{|\mathscr{S}|}\left\{\lambda_{\max}\left[\sum_{i \in [1,p]} h_i^*(N) h_i(N)\right]\right\}^{-1} \leq \tilde{\mathscr{M}}^{-1}(N, \mathscr{S}, \bar{\theta}, u_o(.))$$

$$\leq \frac{|\mathscr{S}_1|}{|\mathscr{S}|}\left\{\max_{i=1,\ldots,p} ||h_i(N)||^2\right\}^{-1},$$

$$\forall \mathscr{S} \in \tilde{\mathscr{P}}(\mathscr{Z}).$$

Corollary 3

For each period $N \in \mathscr{Z}$, the optimal measurement schedule $\mathscr{S}^\phi$ is a *singleton*; up to time delays, the solution of Problem $\tilde{D}$ is precisely given by

$$\mathscr{S}^\phi = \{0\} \tag{56}$$

$$u^\phi(\cdot) = u^o(N, S_1, .). \tag{57}$$

Proof

Theorems 5 and 7 imply that (56), (57) is a solution of Problem $\tilde{D}$. On the other hand, in view of the invariance of system (7)-(9) and of the form of constraints (12), (13), the set $\{h\}$, $h \in S_N$, and the control $u(k) \triangleq u^o(N, S_1, k-h)$ is a solution of Problem $\tilde{D}$ too.

Remark 3

In the case of single input-single output system, Corollary 3 is a trivial consequence of Theorem 6 and Proposition 3.

Remark 4

Obviously, in view of Theorem 1, a solution of Problem D is given by (56) and the periodic extension of (57).

Remark 5

The peculiar form of (56) enables one to solve Problem C for any measurement schedule set $\mathscr{S} \in \mathscr{P}(\mathscr{Z})$ of a given cordinality $\nu \in \mathscr{Z}^{+}$. As a matter of fact, it is apparent that the choice

$$\mathscr{S}_{\phi} = \{0, N, 2N, \ldots, (\nu-1)N\}$$

$$u_{\phi}(.) = PE[u^{\phi}(.)],$$

where PE[.] denotes periodic extension, reults in a maximal "information" for parameter identification. Correspondingly, the minimum parameter estimators variance is given by

$$\frac{1}{\nu}\{\lambda_{max}[\textstyle\sum_{i \in [1,p]} h_i^*(N) h_i(N)]\}^{-1} \quad .$$

Note that such a strong conclusion cannot be drawn in the general case.

6. OPTIMAL PERIOD DESIGN

As far as the period design is concerned, the cases $\mathscr{S}=S_N$ and $\mathscr{S}=S_1$ are here discussed.

With reference to definition (45), denote by $\lambda(\alpha)$ the maximum eigenvalue of $\Pi(\exp(j\alpha))$, $\alpha \in [0,2\pi]$, and by $V(\alpha)$ a unitary norm eigenvalue of $\Pi(\exp(j\alpha))$ correspodning to $\lambda(\alpha)$. Observe that the continuity of the function $\lambda(.)$ implies that there exists $\alpha^o \in [0,2\pi]$ such that

$\lambda(\alpha^o) \geq \lambda(\alpha), \quad \forall\, \alpha \in [0,2\pi]$.

Then, the following result can be stated.

<u>Theorem 8</u>

Let $\mathscr{S} = S_N$.

If $\alpha^o/2\pi$ is a rational number, so that there exist two intergers a^o and N^o such that

$$\frac{\alpha^o}{2\pi} = \frac{a^o}{N^o} ,$$

then a solution of Problem $\tilde{B}$ is given by

$$N^{\theta} = N^o$$

$$u^{\theta}(k) = u^o(N^o, S_{N^o}, k) = \begin{cases} \sqrt{2}\ \mathrm{Re}\{V(a^o)\exp(jka^o \frac{2\pi}{N^o})\} & , \quad a^o = 0 \\ \\ V(0) & , \quad a^o = 0. \end{cases}$$

If $\alpha^o/2\pi$ is an irrational number, for each $N \geq 0$ there exists an $\hat{N} > N$ such that

$$J(\hat{N}, S_{\hat{N}}, u^o(\hat{N}, S_{\hat{N}}, \cdot)) > J(N, S_N, u^o(N, S_N, \cdot))$$

and, therefore, the optimal period does not exist.

<u>Proof</u>

The first part of the theorem immediately follows from Theorem 4. As for the second part, recall that, according to Theorem 4, the optimal N-periodic control is given by a discrete-time sinusoid of frequency h^o/N. Then, since $\alpha^o/2\pi$ is an irrational number

$$\frac{h^o}{N} 2\pi \neq \alpha^o, \qquad \forall N \geq 1.$$

Therefore, letting

$$\gamma \triangleq \min\{\alpha^o, \frac{h^o}{N} 2\pi\}$$

$$\delta \triangleq \max\{\alpha^o, \frac{h^o}{N} 2\pi\},$$

in view of the continuity of $\lambda(\cdot)$, it is possible to find two non-zero integers $\hat{a}$ and $\hat{N}$ such that

$$\frac{\hat{a}}{\hat{N}} \in (\gamma,\delta)$$

and

$$\lambda(\frac{h^o}{N} 2\pi) = J(N,S_N,u^o(N,S_N,.)) < J(\hat{N},S_{\hat{N}},\hat{u}(.)) < \lambda(\alpha^o),$$

where

$$\hat{u}(k) \triangleq \sqrt{2}\,\mathrm{Re}\left\{V(\frac{\hat{a}}{\hat{N}} 2\pi)\ \exp(jk\hat{a}\frac{2\pi}{\hat{N}})\right\}.$$

<u>Theorem 9</u>

Let $\mathscr{S} = S_1$.

Then, for each $N \geq 1$, there exist an integer $\check{N} > N$ such that

$$J(\check{N},S_1,u^o(\check{N},S_1,.)) > J(N,S_1,u^o(N,S_1,.));$$

hence, the optimal period does not exist.

<u>Proof</u>

With reference to Theorem 5, it is

$$J(N,S_1,u^o(N,S_1,.)) = \lambda_{max}[\textstyle\sum_{i \in [1,p]} h_i^*(N)h_i(N)]$$

$$\geq \max_{i=1,\ldots,p} ||h_i(N)||^2 \tag{58}$$

Now, from (50) it follows that

$$h_i(N) = z_i\Phi(N),$$

where z_i is the i-th row of matrix (49) and $\Phi(N)$ has been defined in (48). Since z_i must be non-zero and there exists at least one element of the transfer matrix G(z) which may be zero in at most n-1 points, there exists a $\check{r}$ such that

$$||h_i(\check{r}N)||^2 > ||h_i(N)||^2, \qquad \forall i \in [1,p] \tag{58}$$

From (58) and (59), the statement follows.

$\square$

Theorems 8 and 9 show that the performance index admits a maximum with respect to N neither if $\mathscr{S} = S_N$ and $\alpha^o/2\pi$ is irrational nor if $\mathscr{S}=S_1$. However, only in the latter of these two cases the suprumum of the performance index takes on an infinite value; precisely, the following result can be stated.

Theorem 10

The suprepum of the performance index $J(N,\mathscr{S},u^o(N,\mathscr{S},\cdot))$ with respect to the period N is given by

$$\sup_{N\geq 1} J(N,S_N,u^o(N,S_N,\cdot)) = \lambda(\alpha^o) \tag{60}$$

for $\mathscr{S}=S_N$ and by

$$\sup_{N\geq 1} J(N,S_1,u^o(N,S_1,.)) = +\infty \tag{61}$$

for $\mathscr{S}= S_1$.

Proof

Formula (60) can easily be derived from the proof of Theorem 8.

As for (61), observe first that

$$\sup_{N\geq 1} J(N,S_1,u^o(N,S_1,.)) = \lim_{N\to\infty} J(N,S_1,u^o(N,S_1,.))$$

$$= \lim_{N\to\infty} \lambda_{max}[\Phi^*(N)R^{-1}\Phi(N)]$$

(See Theorems 5 and 9.) Since R^{-1} is a positive definite matrix, (61) is proved by showing that

$$\lim_{N\to\infty} \lambda_{max}[\Phi^*(N)\Phi(N)] = +\infty \ . \tag{62}$$

With this aim, denote by $\phi_i(N)$, the i-th row of $\Phi(N)$.

From the identity

$$\Phi^*(N)\Phi(N) = \sum_{i \in [1,p]} \phi_i^*(N)\phi_i(N),$$

it follows that

$$\lambda_{max}[\Phi^*(N)\Phi(N)] \geq ||\phi_i||^2, \quad \forall i \in [1,p]. \tag{63}$$

Let now t, $t \in [1,p]$, be any integer such that, for some integer $v \in [1,m]$, the transfer function $G_{t,v}(z)$ is not identically zero. From (48), it is apparent that

$$||\phi_t(N)||^2 \geq \sum_{k \in S_N} |G_{t,v}(\exp(jk \tfrac{2\pi}{N}))|^2 \quad . \tag{64}$$

Then, consider an interval $[\beta_1,\beta_2] \subseteq [0,2\pi]$, $\beta_2>\beta_1$, such that

$$G_{t,v}(\exp(j\alpha)) \neq 0, \qquad \forall\alpha \in [\beta_1,\beta_2]$$

(such an interval does obviously exists) and let

$$\mathcal{g} = \min_{\alpha \in [\beta_1,\beta_2]} |G_{t,v}(\exp(j\alpha))|^2. \tag{65}$$

From (64) and (65), it follows that

$$||\phi_t(N)||^2 \geq \mathcal{g}\, \tau(N), \tag{66}$$

where $\tau(N)$ is the cardinality of

$$[\beta_1,\beta_2] \cap \{0 \tfrac{2\pi}{N}, 1 \tfrac{2\pi}{N}, \ldots, (N-1) \tfrac{2\pi}{N}\} \quad .$$

Finally, since $\mathcal{g} > 0$ and

$$\lim_{N\to\infty} \frac{\tau(N)}{N} = \frac{\beta_2-\beta_1}{2\pi}, \tag{67}$$

(63) and (66) give (61) and the proof is completed.

Remark 6

In view of the above result, an interesting comparison between the trends of the approximate variance lower bound for the measurement allocation sets

$$\mathscr{P}' = [0,\ \zeta N-1] \tag{68}$$

$$\mathscr{P}'' = \{0, N, 2N, \ldots, (\zeta N-1)N\},$$

where the cardinality ν of the two sets is the same and is given by $\nu = \zeta N$, can usefully be pointed out.

If $\mathscr{P}=\mathscr{P}'$, (60) implies that, either if N is fixed and $\nu\to\infty$ or ζ is fixed and $\nu\to\infty$, the variance lower bound goes to zero as $1/\nu$. On the other hand, definition (18) and Theorems 3 and 5 enable to derive that

$$\tilde{\mathscr{M}}^{-1}(N, \mathscr{P}'', \bar{\theta}, u_o(.)) = \frac{1}{\zeta N}\{\lambda_{max}[\Phi^*(N)R^{-1}\Phi(N)]\}^{-1}.$$

Therefore, the variance lower bound corresponding to $\mathscr{P}=\mathscr{P}''$ goes to zero as $1/\nu$ for $\nu\to\infty$ and N fixed, while, in view of (63), (64), (66), and (67), for $\nu\to\infty$ and ζ fixed, it goes to zero at least as $1/\nu^2$.

$\square$

For the sake of illustration of the results presented in Sections 4-6, the simple case of single input-single output systems is now briefly discussed.

Remark 7

Consider a single input-single output system; in this case, G(z) is simply the nominal sensitivity system transfer function. Suppose that the measurable output is observed for $\mathscr{P}=\mathscr{P}'$ (see (68)). Then (Theorem 4), the optimal control for the parameter identification is given by

$$u^o(k) = \begin{cases} \sqrt{2}\cos(kh^o \frac{2\pi}{N}), & h^o \neq 0 \\ 1 \qquad\qquad\qquad , & h^o = 0, \end{cases}$$

where h^o is such that

$$|G(\exp(jh^o \frac{2\pi}{N}))| \geq |G(\exp(jh \frac{2\pi}{N})|, \qquad \forall h \in [0,N-1].$$

The approximate variance lower bound is then

$$\tilde{\mathcal{M}}^{-1}(N,\mathcal{S}',\bar{\theta}, PE[u^o(N,S_N,.)] = \frac{R}{\nu|G(\exp(jh^o \frac{2\pi}{N}))|^2} \tag{69}$$

R being the variance of the additive noise.

If $\mathcal{S} = \mathcal{S}''$, according to Theorems 3 and 5, the optimal control turns out to be

$$u^o(k) = \left[\sum_{0}^{N-1}{}_h |G(\exp(jh \frac{2\pi}{N}))|^2\right]^{-1/2} \left\{\sum_{0}^{N-1}{}_h G(\exp(jh \frac{2\pi}{N})) \exp(jkh \frac{2\pi}{N})\right\}$$

and the variance lower bound is given by

$$\tilde{\mathcal{M}}^{-1}(N,\mathcal{S}',\bar{\theta}, PE[u^o(N,S_1,.)]) = \frac{R}{\nu\left[\sum_{0}^{N-1}{}_h |G(\exp(jh \frac{2\pi}{N}))|^2\right]}, \tag{70}$$

which is obviously smaller than (69). Finally, it is apparent that, for a fixed ζ, (69) and (70) go to zero, when ν goes to the infinity, as $1/\nu$ and at least as $1/\nu^2$ respectively.

7. CONCUDING REMARKS

In this paper, a number of problems concerning the optimal cycling for parameter identification of discrete-time and time-invariant systems with output additive noise have been raised and properly stated as Periodic Optimization Problems. As it is natural to expect, explicit solutions are given for linear systems only, affected by a scalar parameter.

The results reported in the present paper leave a number of interesting questions open for further research; among them, the extension to the multiparameter case and/or to problems characterized by non-integral control constraints of the results given in Sections 4 - 6 as well as the same analysis for systems affected by additive noise not only on the output equation but on the state equation too seem to be the most challenging ones.

ACKNOWLEDGEMENT

For most valuable suggestions, the author is grateful to Professor Guido Guardabassi, who also conjectured the result presented here as Corollary 3. For their friendly cooperation, the author wishes to thank Professor Arturo Locatelli and Dr. Nicola Schiavoni, who, together with Professor Guido Guardabassi, made it possible his leave period at the Division of Engineering and Applied Physics, Harvard University. Finally, thanks are also due to Professor Y. C. Ho for his kind hospitality and to a NATO Feelowship, by the Centro di Teoria dei Sistemi (CNR) of Milano (Italy) for partial support.

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